Pergamon

Int. J. Solids Structures Vol. 32, No. 8/9, pp. 1161–1177, 1995

0020-7683/95 \$9.50 + .00

0020–7683(94)00180–4

ASYMPTOTIC ANALYSIS OF NONLINEAR MODE LOCALIZATION IN A CLASS OF COUPLED CONTINUOUS STRUCTURES

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(*Received* 15 *December* 1993; *in revised form* 2 *July* 1994)

Abstract—An asymptotic methodology is developed for studying the localized nonlinear normal modes of a class of conservative periodic continuous structures. Localized modes are synchronous free periodic oscillations which are spatially localized to only a limited number of components of the structure. Nonlinear mode localization is analytically studied by defining modal functions to relate the motion of an arbitrary particle of the structure to the motion of a reference point. Conservation of energy is imposed to construct a set of singular nonlinear partial differential equations governing the modal functions. This set is asymptotically solved using a perturbation technique. Criteria for existence of strongly or weakly localized modes are formulated, and are subsequently used for detecting nonlinear mode localization in the system. The orbital stability of the detected localized modes is then investigated by expanding the corresponding variational equations in bases of orthogonal polynomials and using Floquet theory to analyze the resulting set of linear differential equations with periodic coefficients. Application of the general theory is given by studying the localized modes of a system of two weakly coupled, simply supported beams resting on nonlinear elastic foundations. The use of mode localization for the active or passive spatial confinement of impulsive responses of flexible structures is discussed.

1. INTRODUCTION

In recent studies, linear and nonlinear mode localization in periodic or nearly periodic structures was analytically and numerically investigated. It was theoretically shown that, under certain conditions, the spatial distributions of a subset of modes of a periodic structure can become spatially localized to only a limited number of substructures. An interesting feature of structures with localized modes is that they possess *passive motion confinement properties*—motions generated by external transient impulses remain passively confined close to the point of application of the disturbances, instead of "spreading" through the entire structure (Cornwell and Bendiksen, 1989; Vakakis, 1993). This passive motion confinement phenomenon can be implemented in active or passive vibration and shock isolation designs.

In Pierre (1990), Lust *et al.* (1991), Levine and Salama (1993), and Bouzit and Pierre (1993) the localized modes of coupled flexible structures of various configurations were studied by perturbation analysis or numerical finite-element and Rayleigh–Ritz techniques. It was shown that *linear mode localization takes place only in weakly coupled and weakly disordered periodic flexible structures.* Experimental verification of mode localization in disordered linear periodic structures was carried out in Levine and Salama (1993) and Pierre (1990). Nonlinear mode localization in flexible structures was analytically and numerically studied in King and Vakakis (1993a), and Vakakis (1993). In these studies the notion of nonlinear normal mode (Rosenberg, 1966; Rand, 1971b; Vakakis, 1990) was employed, and Gallerkin approximations were performed to descretize the governing nonlinear partial differential equations of motion, thereby converting the infinite-dimensional continuous problem to a finite-dimensional one. Approximate perturbation techniques were then employed to compute the localized nonlinear normal modes of the discretized problem. It was found that *nonlinear mode localization takes place in weakly coupled flexible structures even in the absence of structural disorder.* In fact, the principal mechanism for nonlinear mode localization was proved to be the nonlinear relation between amplitude and frequency during nonlinear free oscillation (Vakakis and Cetinkaya, 1992; Vakakis *et al.*, 1993).

In studies of nonlinear free oscillations employing Gallerkin approximations, the structural responses are expressed as superpositions of linearized eigenfunctions (Nayfeh *et al.*, 1974). Although these techniques provide nonlinear corrections to the modal amplitudes of the superimposed linearized eigenfunctions, they do not provide any nonlinear corrections to the mode shapes of the ensuing nonlinear free oscillations. This is a drawback of Gallerkin-type nonlinear analyses. In King and Vakakis (1993b), a new asymptotic method for studying the free oscillations of single nonlinear structural components was developed. The method was based on the nonlinear separation of the spatial and temporal variables in the governing partial differential equation of motion, and provided nonlinear corrections to the mode shape of the structure during free oscillation. In the aforementioned work the method was employed to study the free oscillations of simply supported and cantilever beams, with geometric nonlinearities due to finite-amplitude motions.

In this work a new method for studying the localized free oscillations of a class of coupled conservative nonlinear flexible structures is developed. The method does not attempt discretization of the equations of motion, and can be regarded as an extension of the technique first developed in King and Vakakis (1993b). Modal functions are introduced to express the motion of an arbitrary particle of the structural assembly in terms of the motion of a reference point. Conservation of energy is then imposed to construct the set of singular partial differential equations governing the modal functions. The spatially localized solutions of this set of equations are asymptotically approximated using series expansions. The stability of the detected localized free oscillations is investigated by expanding the corresponding variational equations in bases of orthogonal polynomials and using Floquet analysis. The developed technique is employed to the study of nonlinear mode localization in an assembly of two coupled beams resting on nonlinear elastic foundations.

2. THEORETICAL DEVELOPMENT

Consider the free vibration of a periodic structural assembly composed of N identical one-dimensional continuous substructures, which are coupled by means of elastic elements. Since all substructures are assumed to possess identical geometrical characteristics, the equilibrium position of each substructure is parametrized by the same normalized spatial variable s, $0 \leqslant s \leqslant 1$. Displacements from equilibrium of material points of the ith substructure are denoted by variables $u_i(s, t)$, $i = 1, \ldots, N$. It is assumed that the free motions of the periodic assembly are governed by a set of N nonlinear partial differential equations of the following form:

$$u_{itt} = \mathbf{L}_i[u_1(s, t), \ldots, u_N(s, t); \varepsilon], \quad s \in [0, 1], \quad i = 1, \ldots, N. \tag{1}$$

These equations are complemented by the set of homogeneous boundary conditions

$$\mathbf{B}_i[u_1(s, t), \ldots, u_N(s, t); \varepsilon] = 0, \quad s = 0, 1, \quad i = 1, \ldots, N. \tag{2}$$

$\mathbf{L}_i[\cdot]$ is an integro-differential operator acting on variables $u_p(s, t)$, $p = 1, \ldots, N$, whereas $\mathbf{B}_i[\cdot]$ is a boundary condition operator. The nonlinear terms in operators $\mathbf{L}_i[\cdot]$ and $\mathbf{B}_i[\cdot]$ are assumed to be small and proportional to a small parameter ε, $|\varepsilon| \ll 1$. The following assumptions are now imposed on the boundary value problem (1)–(2):

(i) the total energy of system (1) is conserved, and boundary conditions (2) involve no dissipation of energy;

(ii) for $\varepsilon = 0$, system (1)–(2) is separable in space and time, and admits bounded periodic solutions for any set of initial conditions;

(iii) operators $\mathbf{L}_i[\cdot]$ involve integro-differential operations in terms of the spatial variable;

(iv) only odd-order nonlinearities are present in the system;

(v) the displacements $u_i(s, t)$ are sufficiently smooth functions of their variables, so that all derivatives appearing in the following analysis exist.

Assumption (i) is needed in order to impose the condition of preservation of the total energy of oscillation, whereas (ii)–(v) are necessary for carrying the perturbation analysis that follows. Condition (iv) guarantees a symmetric structure for the computed localized free oscillations.

As in Rosenberg (1966) and King and Vakakis (1993b), one defines *nonlinear normal modes* of system (1)–(2) as synchronous free oscillations, where all material points of the periodic assembly vary equiperiodically, reaching their extreme values at the same instant of time. Assuming that the assembly is oscillating in a nonlinear normal mode, the displacement of the kth substructure at position $s = s_o$ is denoted by $u_o(t) \equiv u_k(s_o, t)$. The kth substructure will be termed *reference substructure*, and the quantity $u_o(t)$ *reference displacement*. Since the free oscillation is assumed to be synchronous, the displacement of an arbitrary material point of an arbitrary substructure can be parameterized in terms of the reference displacement $u_o(t)$ as follows:

$$u_i(s,t) = U_i(s, u_o(t)), \quad i = 1, \ldots, N, \quad U_k(s_o, u_o(t)) \equiv u_o(t). \tag{3}$$

Function $U_i(\cdot,\cdot)$ is termed the *ith modal function*, and describes the mode shape of the ith substructure during a nonlinear normal mode of the structural assembly. In writing eqn (3), the set of independent variables is changed from (s, t) to $(s, u_o(t))$, and *the explicit time dependence of the motion is eliminated from the expression of* u_i. This transformation of coordinates requires s_o not to coincide with a node of the nonlinear mode under investigation. Shaw and Pierre (1992) used a similar functional relation to define the nonlinear mode shape, using both the displacement *and* the velocity of the reference point as new independent variables.

Considering relations (3), one formulates a condition for *strong nonlinear mode localization* in the periodic assembly by requiring that the displacement of point s^* of the kth (reference) substructure be much larger in magnitude than the corresponding point of any other substructure of the system:

$$|U_k(s^*, u_o(t))| \gg |U_i(s^*, u_o(t))|, \quad i = 1, \ldots, N, i \neq k, \quad 0 \leqslant s^* \leqslant 1,$$
$$\forall t \geqslant 0, \quad U_k(s^*, u_o(t)) \neq 0 \quad \text{(strong nonlinear mode localization)}. \tag{4}$$

Conditions (4) are imposed away from nodes of the nonlinear normal mode. These conditions guarantee spatial confinement of the free oscillation to the kth substructure. Conditions for *weak nonlinear mode localization* can be formulated in a similar way. To this end, one requires the nonlinear oscillation to be mainly confined to a subset of substructures (instead of only the reference substructure):

$$|U_m(s^*, u_o(t))| \gg |U_i(s^*, u_o(t))|, \quad m \in S_1, \quad i \notin S_1, \quad [1, \ldots, N] \supset S_1, \quad 0 \leqslant s^* \leqslant 1,$$
$$\forall t \geqslant 0, \quad U_m(s^*, u_o(t)) \neq 0 \quad \text{(weak nonlinear mode localization)}. \tag{5}$$

Previous works demonstrated the existence of strong and weak localization in weakly coupled nonlinear periodic assemblies (Vakakis *et al.*, 1993; Vakakis and Cetinkaya, 1992; Vakakis, 1993). However, in these works the systems considered were either discrete or discretized using Gallerkin approximations. In what follows, a new asymptotic methodology for studying nonlinear mode localization in continuous periodic assemblies is described. The method relies on the analytical evaluation of the modal functions $U_i(s, u_o(t))$ by direct analysis of the governing nonlinear partial differential equations of motion without resorting to any discretization scheme. Once analytic expressions for $U_i(s, u_o(t))$ are computed, criteria (4) and (5) are used to detect nonlinear mode localization in the periodic assembly.

Since the total energy, E_{tot}, during free oscillation is conserved, one formulates an energy conservation relation of the form:

$$E_{tot} = \sum_{i=1}^{N} \left[(1/2) \int_0^1 u_{it}^2 \, ds + L_i[u_1(s,t), \ldots, u_N(s,t); \varepsilon] \right], \tag{6}$$

where $L_i[\cdot]$ is a spatial operator acting on the displacements; this operator represents potential energy terms arising from operator $\mathbf{L}_i[\cdot]$ in eqn (1). Typically, $L_i[\cdot]$ involves integrations with respect to the spatial variable, with integrands containing partial derivatives with respect to the displacements. Since the periodic assembly under consideration is conservative, relation (6) holds at all times. In what follows, relations (3) and the energy equality (6) are used to eliminate the time derivatives from the equation of motions, and to obtain the equations governing the modal functions $U_i(\cdot,.)$.

Employing relations (3), the partial derivatives of the displacements with respect to time are expressed as follows:

$$u_{it} = (\partial U_i/\partial u_o)\, u_{ot}, \quad u_{itt} = (\partial^2 U_i/\partial u_o^2)\, u_{ot}^2 + (\partial U_i/\partial u_o)\, u_{ott}, \ldots, \tag{7}$$

where subscripts t denote derivatives with respect to the time variable. Using eqn (7), the energy conservation relation is written as

$$E_{tot} = \sum_{i=1}^{N} \left[(1/2)\, u_{ot}^2 \int_0^1 (\partial U_i/\partial u_o)^2 \, ds + L_i[U_1, \ldots, U_N; \varepsilon] \right], \tag{8}$$

where the integro-differential operator L_i is derived from L_i by substituting $u_p(s,t) = U_p(s, u_o(t))$, $p = 1, \ldots, N$. Using relation (8), one can express the velocity of the reference point, u_{ot}, in terms of the modal functions U_i and the total energy E_{tot}:

$$u_{ot}^2 = 2\left[E_{tot} - \sum_{i=1}^{N} L_i[U_1, \ldots, U_N; \varepsilon] \right] \left[\sum_{i=1}^{N} \int_0^1 (\partial U_i/\partial u_o)^2 \, ds \right]^{-1}. \tag{9}$$

Expressions (3) and (9) evaluate the displacement and velocity of the reference point in terms of the (yet unknown) modal functions $U_p(s_i, u_o(t))$. Substituting eqns (3) and (7) into the equations of motion (1), taking into account eqn (9), and noting that the acceleration of the reference point is computed by $u_{ott} = \mathbf{L}_k[U_1(s_o, t), \ldots, U_N(s_o, t); \varepsilon]$, one obtains the following equations governing the modal functions $U_i(s, u_o(t))$:

$$2\left[E_{tot} - \sum_{i=1}^{N} L_i[U_1, \ldots, U_N; \varepsilon] \right] \left[\sum_{i=1}^{N} \int_0^1 (\partial U_i/\partial u_o)^2 \, ds_i \right]^{-1} \frac{\partial^2 U_i}{\partial u_o^2}$$
$$+ \underline{\mathbf{L}}_k[U_1(s_o, u_o), \ldots, U_N(s_o, u_o); \varepsilon] \frac{\partial U_i}{\partial u_o} = \underline{\mathbf{L}}_i[U_1(s, u_o), \ldots, U_N(s, u_o); \varepsilon], s \in [0, 1], \tag{10}$$

where $i = 1, \ldots, N$. The following boundary conditions complement eqn (10):

$$\underline{\mathbf{B}}_i[U_1(s, u_o(t)), \ldots, U_N(s, u_o(t))] = 0, \quad s = 0, 1, \quad i = 1, \ldots N. \tag{11}$$

In eqns (10) and (11), operators $\underline{\mathbf{L}}_i[\cdot]$ and $\underline{\mathbf{B}}_i[\cdot]$ are derived from the original operators $\mathbf{L}_i[\cdot]$ and $\mathbf{B}_i[\cdot]$ of eqns (1)–(2) by transforming the set of independent variables from (s, t) to $(s, u_o(t))$. Denoting by u_o^* the maximum amplitude attained by $u_o(t)$ during free oscillation, and taking into account that the motion of the system is synchronous, it is noted *that equations* (10) *become singular when the periodic assembly reaches its maximum potential energy level*, i.e. when $u_o(t) = u_o^*$ and

$$E_{tot} = \left\{ \sum_{i=1}^{N} L_i[U_1, \ldots, U_N; \varepsilon] \right\}_{u_o = u_o^*}.$$

This is due to the fact that, at maximum potential energy, the coefficients of the highest derivatives ($\partial^2 U_i/\partial u_o^2$) in eqn (10) vanish. Therefore, for values of the potential energy less than E_{tot}, asymptotic approximations to the functions U_i are constructed, and then analytically continued up to the maximum potential energy level E_{tot}. Similar asymptotic analyses for discrete oscillators were carried out by Rand (1971a), Manevich and Mikhlin, (1972), Mikhlin (1974), Vakakis (1992), and for continuous oscillators by King and Vakakis (1993b). Analytic continuations of the asymptotic solutions up to the maximum potential energy level are achieved by complementing eqns (10) with the following set of relations which hold at the maximum potential energy level:

$$\left[\mathbf{L_k}[U_1(s_o, u_o), \ldots, U_N(s_o, u_o); \varepsilon] \frac{\partial U_i}{\partial u_o} - \mathbf{L_i}[U_1(s, u_o), \ldots, U_N(s, u_o); \varepsilon] \right]_{u_o = u_o^*} = 0$$

$$i = 1, \ldots, N. \quad (12)$$

Summarizing, the normal modal shapes $U_i(s, u_o(t))$ are governed by the set of singular nonlinear partial differential equations (10), subjected to the boundary conditions (11) and the maximum potential energy relations (12). Asymptotic approximations to the solutions of equations (10)–(12) are sought in the form of a power series. Away from the maximum potential energy level, the solution of eqns (10) and (11) are represented in the series form:

$$U_i(s, u_o(t)) = \sum_{j=0}^{\infty} U_i^{(j)}(s, u_o), \quad i = 1, \ldots, N. \quad (13)$$

It is assumed that functions $U_i^{(j)}$ and all their partial derivatives are of order ε^j:

$$\frac{\partial^n U_i^{(j)}}{\partial s^n} = O(\varepsilon^j), \quad j = 0, 1, \ldots \quad (14)$$

Hence, the quantities $U_i^{(j)}$ are the $O(\varepsilon^j)$ corrections to the ith modal amplitudes U_i. Substituting eqn (13) into eqns (10)–(12) and matching coefficients of respective powers of ε, one obtains a series of subproblems governing the modal corrections $U_i^{(j)}$. These subproblems are then solved by expressing $U_i^{(j)}$ in power expansions of the form

$$U_i^{(0)}(s, u_o) = a_{i1}^{(0)}(s)\, u_o(t), \quad U_i^{(j)}(s, u_o) = \sum_m a_{im}^{(j)}(s)\, u_o^m(t), \quad j = 1, 2, \ldots \quad (15)$$

The expressions for the $O(1)$ terms $U_i^{(0)}(s, u_o)$ are separable in space and time due to assumption (ii) of the problem. The spatial coefficients $a_{im}^{(j)}(s)$ are computed by substituting eqn (15) into the $O(\varepsilon^j)$ subproblem governing $U_i^{(j)}(s, u_o)$ and matching coefficients of respective powers of $u_o(t)$. A set of ordinary differential equations governing $a_{im}^{(j)}(s)$ is then derived, which, coupled with appropriate boundary conditions, is solved to provide the spatial coefficients in eqn (15). The described analytic methodology can be carried out up to any desired order of approximation. Once analytic approximations for the modal functions U_i are obtained, the time response $u_o(t)$ of the reference point is analytically approximated by considering the kth equation of motion at the reference point $s = s_o$:

$$u_{ott} = \mathbf{L_k}[U_1(s_o, u_o(t)), \ldots, U_N(s_o, u_o(t)); \varepsilon]. \quad (16)$$

For a prescribed set of initial conditions $u_o(0)$ and $u_{ot}(0)$, eqn (16) can be solved either by quadratures or by an approximate nonlinear perturbation technique (Nayfeh and Mook, 1984). Once $u_o(t)$ is determined, the oscillation of an arbitrary point of the periodic assembly can be computed using relations (3).

If a subset of the computed modal functions $U_i(s, u_o(t))$ satisfies criteria (4) or (5), mode localization occurs in the structure. Criteria (4) and (5) guarantee the existence of strong or weak nonlinear mode localization in a *mathematical sense*. However, as past works have shown (Vakakis *et al.*, 1993), not all mathematically computed nonlinear localized modes are physically realizable, since some may be *orbitally unstable*. To determine if the computed localized modes are physically attainable, one performs a numerical stability analysis based on Floquet theory. The stability analysis is described in detail in the next section, where a specific application of the outlined asymptotic methodology is given.

3. APPLICATION: COUPLED BEAMS ON NONLINEAR FOUNDATIONS

Application of the asymptotic methodology will be given by studying the localized nonlinear modes of the structure depicted in Fig. 1. This structure consists of two coupled simply supported beams resting on nonlinear elastic foundations. Assuming that the distributed linear coupling stiffness and the nonlinearities of the elastic foundations are weak [of $O(\varepsilon)$, $|\varepsilon| \ll 1$], the governing equations of motion are written in the following form:

$$u_{1tt} = -u_{1ssss} - ku_1 - \varepsilon\gamma u_1^3 - \varepsilon K(u_1 - u_2) \tag{17}$$

$$u_{2tt} = -u_{2ssss} - ku_2 - \varepsilon\gamma u_2^3 - \varepsilon K(u_2 - u_1). \tag{18}$$

Assuming simply-supported beams, the boundary conditions of the problem are given by:

$$\begin{aligned} u_1(0,t) &= u_{1ss}(0,t) = u_1(1,t) = u_{1ss}(1,t) = 0 \\ u_2(0,t) &= u_{2ss}(0,t) = u_2(1,t) = u_{2ss}(1,t) = 0. \end{aligned} \tag{19}$$

Equations (17)–(19) correspond to the general equations (1) and (2) of the previous section. The localized nonlinear normal modes of system (17)–(19) are now sought. Following the previously outlined general methodology, it is assumed that $u_1(s,t) = U_1(s, u_o(t))$, $u_2(s,t) = U_2(s, u_o(t))$, where $U_1(s_o, u_o(t)) \equiv u_o(t)$ is the displacement of the reference point $s = s_o$ of beam 1. It is assumed that the reference point does not coincide with any node of the nonlinear mode under investigation. The energy of the system during the nonlinear mode oscillation is expressed as

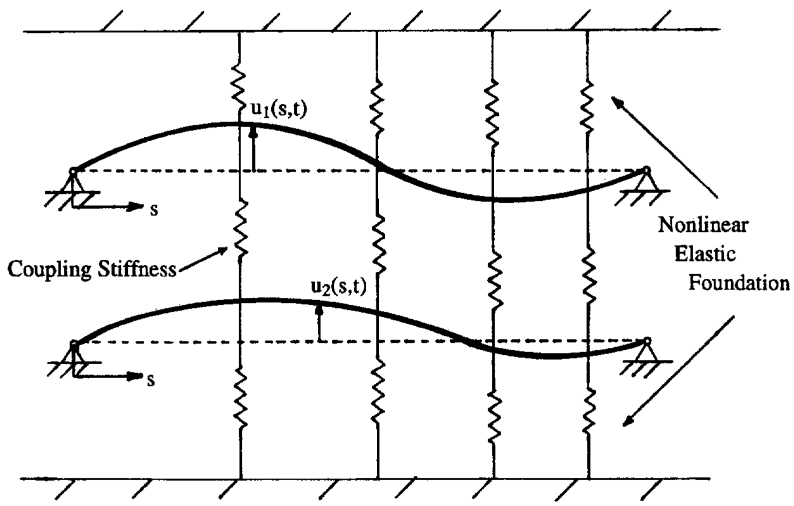

Fig. 1. The flexible system under consideration.

$$E_{tot} = (1/2)\sum_{i=1}^{2}\int_0^1 \left(u_{it}^2 + u_{iss}^2 + ku_i^2 + \frac{\varepsilon\gamma}{2}u_i^4 + \frac{\varepsilon K}{2}(u_i - u_{i+1})^2\right) ds \tag{20}$$

with the understanding that $u_3 \equiv u_1$. The velocity of the reference point is computed as

$$(u_{ot})^2 = \frac{2E_{tot} - \sum_{i=1}^{2}\int_0^1 \left(U_{iss}^2 + kU_i^2 + \frac{\varepsilon\gamma}{2}U_i^4 + \frac{\varepsilon K}{2}(U_i - U_{i+1})^2\right) ds}{\sum_{i=1}^{2}\int_0^1 (\partial U_i/\partial u_o)^2\, ds} \tag{21}$$

with $U_3 \equiv U_1$. Expressing the displacements in the equations of motion in terms of the modal functions, computing $(u_{ot})^2$ from eqn (21), and taking into account that

$$u_{ott} = \{-U_{1ssss} - kU_1 - \varepsilon\gamma U_1^3 - \varepsilon K(U_1 - U_2)\}_{s=s_o},$$

one obtains the following set of singular partial differential equations governing the modal functions $U_1(s, u_o)$ and $U_2(s, u_o)$:

$$\frac{2E_{tot} - \sum_{i=1}^{2}\int_0^1 \left(U_{iss}^2 + kU_i^2 + \frac{\varepsilon\gamma}{2}U_i^4 + \frac{\varepsilon K}{2}(U_i - U_{i+1})^2\right) ds}{\sum_{i=1}^{2}\int_0^1 (\partial U_i/\partial u_o)^2\, ds}\frac{\partial^2 U_1}{\partial u_o^2}$$

$$+ \{-U_{1ssss} - kU_1 - \varepsilon\gamma U_1^3 - \varepsilon K(U_1 - U_2)\}_{s=s_o}\frac{\partial U_1}{\partial u_o} = -U_{1ssss} - kU_1 - \varepsilon\gamma U_1^3 - \varepsilon K(U_1 - U_2)$$

and

$$\frac{2E_{tot} - \sum_{i=1}^{2}\int_0^1 \left(U_{iss}^2 + kU_i^2 + \frac{\varepsilon\gamma}{2}U_i^4 + \frac{\varepsilon K}{2}(U_i - U_{i+1})^2\right) ds}{\sum_{i=1}^{2}\int_0^1 (\partial U_i/\partial u_o)^2\, ds}\frac{\partial^2 U_2}{\partial u_o^2}$$

$$+ \{-U_{1ssss} - kU_1 - \varepsilon\gamma U_1^3 - \varepsilon K(U_1 - U_2)\}_{s=s_o}\frac{\partial U_2}{\partial u_o} = -U_{2ssss} - kU_2 - \varepsilon\gamma U_2^3 - \varepsilon K(U_2 - U_1). \tag{22}$$

These equations are complemented by the following relations holding at maximum potential energy level, $u_o(t) = u_o^*$:

$$\left[\{-U_{1ssss} - kU_1 - \varepsilon\gamma U_1^3 - \varepsilon K(U_1 - U_2)\}_{s=s_o}\frac{\partial U_1}{\partial u_o} + U_{1ssss} + kU_1 + \varepsilon\gamma U_1^3 + \varepsilon K(U_1 - U_2)\right]_{u_o(t)=u_o^*} = 0$$

$$\left[\{-U_{1ssss} - kU_1 - \varepsilon\gamma U_1^3 - \varepsilon K(U_1 - U_2)\}_{s=s_o}\frac{\partial U_2}{\partial u_o} + U_{2ssss} + kU_2 + \varepsilon\gamma U_2^3 + \varepsilon K(U_2 - U_1)\right]_{u_o(t)=u_o^*} = 0. \tag{23}$$

Equations (22) and (23) are solved, taking into account boundary conditions (18). These boundary equations are expressed in terms of the modal functions

$$U_1(0,u_o) = U_{1ss}(0,u_o) = U_1(1,u_o) = U_{1ss}(1,u_o) = 0$$
$$U_2(0,u_o) = U_{2ss}(0,u_o) = U_2(1,u_o) = U_{2ss}(1,u_o) = 0 \qquad (24)$$

and the following compatibility condition is imposed:

$$U_1(s_o, u_o(t)) \equiv u_o(t). \qquad (25)$$

Equations (22) and (23) correspond to eqns (10) and (12) of the previous section. When the potential energy is less than E_{tot}, solutions to eqns (22,24) are sought in the series form (13)–(15). The power series solutions are then analytically continued up to the maximum potential energy level, E_{tot}, by satisfying relations (23).

The modal functions are expressed in the series forms

$$U_i(s,u_o(t)) = U_i^{(0)}(s,u_o) + U_i^{(1)}(s,u_o) + U_i^{(2)}(s,u_o) + O(\varepsilon^3), \quad i = 1,2,$$

and are substituted into eqns (22)–(25). In accordance with assumption (ii) of the general formulation, the $O(1)$ approximations are separable in space and time, $U_i^{(0)}(s,u_o) = a_{i1}^{(0)}(s)\,u_o(t)$. By matching coefficients of respective powers of ε, a series of subproblems at various orders of ε are formulated. In this work the symbolic manipulations required for computing the asymptotic approximations to the modal functions were performed using the software package *Mathematica*.

O(1) *approximation*

Retaining only $O(1)$ terms in eqns (22)–(25), one obtains the following equation governing the first-order corrections $U_i^{(0)}(s,u_o)$:

$$\{-U_{1ssss}^{(0)} - kU_1^{(0)}\}_{s=s_o}\frac{\partial U_1^{(0)}}{\partial u_o} = -U_{1ssss}^{(0)} - kU_1^{(0)}$$
$$\{-U_{1ssss}^{(0)} - kU_1^{(0)}\}_{s=s_o}\frac{\partial U_2^{(0)}}{\partial u_o} = -U_{2ssss}^{(0)} - kU_2^{(0)}. \qquad (26)$$

Setting $U_i^{(0)}(s,u_o) = a_{i1}^{(0)}(s)\,u_o(t)$, $i = 1,2$, and substituting into eqn (26), the following ordinary differential equations governing $a_{i1}^{(0)}(s)$ are derived:

$$a_{11}^{(0)''''}(s) - a_{11}^{(0)''''}(s_o)a_{11}^{(0)}(s) = 0 \Rightarrow a_{11}^{(0)}(s) = A\sin\lambda s + B\cos\lambda s + C\sinh\lambda s + D\cosh\lambda s$$
$$a_{21}^{(0)''''}(s) - a_{11}^{(0)''''}(s_o)a_{21}^{(0)}(s) = 0 \Rightarrow a_{21}^{(0)}(s) = E\sin\lambda s + F\cos\lambda s + G\sinh\lambda s + H\cosh\lambda s \qquad (27)$$

where $\lambda^4 = a_{11}^{(0)''''}(s_o)$ is a solution-dependent parameter, and primes denote differentiation with respect to the spatial variable s. Imposing the boundary conditions, $a_{i1}^{(0)}(0) = a_{i1}^{(0)}(1) = a_{i1}^{(0)''}(0) = a_{i1}^{(0)''}(1) = 0$, $i = 1,2$, and the compatibility relation $a_{11}^{(0)}(s_o) = 1$, the solutions of eqn (27) are computed as

$$a_{11}^{(0)}(s) = \sin(n\pi s)/\sin(n\pi s_o), \quad a_{21}^{(0)}(s) = a_{21}^{(0)}(s_o)\sin(n\pi s)/\sin(n\pi s_o), \quad \lambda = n\pi, \qquad (28)$$

where $n = 1,2,\ldots$ denotes the order of the mode. The amplitude of beam 1 (the reference beam) is completely determined at this order of approximation, whereas the amplitude of the non-reference beam 2 is determined within a multiplicative constant, $a_{21}^{(0)}(s_o)$, which is evaluated at the next order of approximation.

$O(\varepsilon)$ *approximation*

Substituting the series expressions for the modal functions in eqns (22)–(25), and retaining $O(\varepsilon)$ terms, one obtains the following equations governing the first order corrections $U_1^{(1)}(s,u_o)$ and $U_2^{(1)}(s,u_o)$:

$$\frac{2E_{tot}-\sum_{i=1}^{2}\int_0^1 (U_{i\,ss}^{(0)\,2}+kU_i^{(0)2})\,ds}{\sum_{i=1}^{2}\int_0^1 (\partial U_i^{(0)}/\partial u_o)^2\,ds}\frac{\partial^2 U_j^{(1)}}{\partial u_o^2}$$
$$+\{-U_{1ssss}^{(1)}-kU_1^{(1)}-\gamma U_1^{(0)3}-K(U_1^{(0)}-U_2^{(0)})\}_{s=s_o}\frac{\partial U_j^{(0)}}{\partial u_o}$$
$$+\{-U_{1ssss}^{(0)}-kU_1^{(0)}\}_{s=s_o}\frac{\partial U_j^{(1)}}{\partial u_o}$$
$$= -U_{jssss}^{(1)}-kU_j^{(1)}-\gamma U_j^{(0)3}-K(U_j^{(0)}-U_{j+1}^{(0)}),\quad j=1,2 \tag{29}$$

with the understanding that $U_3^{(0)}\equiv U_1^{(0)}$. These equations are complemented by the following relations holding at the maximum potential energy level, $u_o(t)=u_o^*$:

$$\Bigg[\{-U_{1ssss}^{(1)}-kU_1^{(1)}-\gamma U_1^{(0)3}-K(U_1^{(0)}-U_2^{(0)})\}_{s=s_o}\frac{\partial U_j^{(0)}}{\partial u_o}$$
$$+\{-U_{1ssss}^{(0)}-kU_1^{(0)}\}_{s=s_o}\frac{\partial U_j^{(1)}}{\partial u_o}+U_{jssss}^{(1)}+kU_j^{(1)}+\gamma U_j^{(0)3}+K(U_j^{(0)}-U_{j+1}^{(0)})\Bigg]_{u_o(t)=u_o^*}=0 \tag{30}$$

where $j=1,2$. The maximum energy E_{tot} in eqn (29) is equal to the maximum potential energy level, and, hence, can be expressed in terms of the maximum displacement of the reference point u_o^*:

$$E_{tot}=\{(1/2)\sum_{i=1}^{2}\int_0^1 (U_{i\,ss}^{(0)\,2}+kU_i^{(0)2})\,ds\}_{u_o(t)=u_o^*}+O(\varepsilon). \tag{31}$$

The corrections to the modal amplitudes $U_i^{(1)}(s,u_o(t))$ are expressed in the series form,

$$U_i^{(1)}(s,u_o(t))=a_{i1}^{(1)}(s)\,u_o(t)+a_{i3}^{(1)}(s)\,u_o^3(t)+O(u_o^5(t)),\quad i=1,2. \tag{32}$$

Although the analysis can be carried to any order of approximation, in this work only terms up to $O(u_o^3(t))$ are retained in the expressions for $U_i^{(1)}$. Substituting eqn (32) into eqns (29) and (30), taking into account eqn (31), and matching coefficients of respective powers of $u_0(t)$, one obtains the following ordinary differential equations governing the spatial distributions $a_{i1}^{(1)}(s)$ and $a_{i3}^{(1)}(s)$:

$$6(n^4\pi^4+k)u_o^{*2}a_{13}^{(1)}(s)-[a_{11}^{(1)''''}(s_o)]\frac{\sin(n\pi s)}{\sin(n\pi s_o)}-n^4\pi^4a_{11}^{(1)}(s)=-a_{11}^{(1)''''}(s)$$
$$6(n^4\pi^4+k)u_o^{*2}a_{23}^{(1)}(s)-[a_{11}^{(1)''''}(s_o)]a_{21}^{(0)}(s_o)\frac{\sin(n\pi s)}{\sin(n\pi s_o)}-n^4\pi^4a_{21}^{(1)}(s)$$
$$=-a_{21}^{(1)''''}(s)+K(1-a_{21}^{(0)}(s_o))(1+a_{21}^{(0)}(s_o))\frac{\sin(n\pi s)}{\sin(n\pi s_o)}. \tag{33}$$

Equation (33) evaluates the cubic spatial coefficients $a_{i3}^{(1)}(s)$ in terms of the (yet unknown) linear spatial coefficients $a_{i1}^{(1)}(s)$. These coefficients are governed by the following eighth-order differential equations:

$$a_{11}^{(1)(\mathrm{viii})}(s)-(10n^4\pi^4+8k)a_{11}^{(1)(\mathrm{iv})}(s)+n^4\pi^4(9n^4\pi^4+8k)a_{11}^{(1)}(s)$$
$$=\left[\frac{\sin(n\pi s)}{\sin(n\pi s_o)}\{a_{11}^{(1)(\mathrm{viii})}(s_o)-(10n^4\pi^4+8k)a_{11}^{(1)(\mathrm{iv})}(s_o)-6\gamma(n^4\pi^4+k)u_o^{*2}\}\right.$$
$$\left.+6\gamma(n^4\pi^4+k)u_o^{*2}\frac{\sin^3(n\pi s)}{\sin^3(n\pi s_o)}\right] \quad (34)$$

and

$$a_{21}^{(1)(\mathrm{viii})}(s)-(10n^4\pi^4+8k)a_{21}^{(1)(\mathrm{iv})}(s)+n^4\pi^4(9n^4\pi^4+8k)a_{21}^{(1)}(s)$$
$$=-2(n^4\pi^4+k)[3K(1-a_{21}^{(0)}(s_o))+3a_{21}^{(0)}(s_o)\{a_{11}^{(1)(\mathrm{iv})}(s_o)+u_o^{*2}a_{13}^{(1)(\mathrm{iv})}(s_o)+\gamma u_o^{*2}$$
$$+K(1-a_{21}^{(0)}(s_o))\}+a_{11}^{(1)(\mathrm{iv})}(s_o)a_{21}^{(0)}(s_o)+K(1-a_{21}^{(0)2}(s_o))]\frac{\sin(n\pi s)}{\sin(n\pi s_o)}$$
$$+[6\gamma u_o^{*2}(n^4\pi^4+k)a_{21}^{(0)3}(s_o)]\frac{\sin^3(n\pi s)}{\sin^3(n\pi s_o)}. \quad (35)$$

The nonhomogeneous terms in eqns (34)–(35) are *solution-dependant*, since they contain the (yet unknown) first approximation $a_{21}^{(0)}(s_o)$, and derivatives of $a_{11}^{(1)}(s)$ evaluated at s_0. Moreover, these equations depend on the order n of the mode under examination. Equations (34)–(35) are complemented by the boundary conditions,

$$\frac{d^p a_{i1}^{(1)}(s)}{ds^p}_{s=0}=\frac{d^p a_{i1}^{(1)}(s)}{ds^p}_{s=1}=0,$$

for $p = 0, 2, 4, 6$, and $i = 1, 2$, and by the compatibility relations, $a_{11}^{(1)}(s_o) = a_{13}^{(1)}(s_o) = 0$. These conditions were derived by considering $O(\varepsilon)$ terms in expressions (24) and (25).

The solutions of eqns (34) and (35) are composed of homogeneous and particular solution terms, and their derivation is standard. Equation (34) coupled with the boundary conditions and compatibility relation yields the following solution for $a_{11}^{(1)}(s)$:

$$a_{11}^{(1)}(s)=\frac{3\gamma(n^4\pi^4+k)u_o^{*2}}{320n^4\pi^4(9n^4\pi^4-k)}\left[\frac{\sin^3(n\pi s)}{\sin^3(n\pi s_o)}-\frac{\sin(n\pi s)}{\sin(n\pi s_o)}\right]. \quad (36)$$

Substituting eqn (36) into the first of equations (33) leads to the following solution for the cubic spatial coefficient of the reference beam:

$$a_{13}^{(1)}(s)=\frac{\gamma}{8(9n^4\pi^4-k)}\left[\frac{\sin(n\pi s)}{\sin(n\pi s_o)}-\frac{\sin^3(n\pi s)}{\sin^3(n\pi s_o)}\right]. \quad (37)$$

Considering eqns (36) and (37), it is noted that they become singular when $k = 9n^4\pi^4$. The same finding was reported by Shaw and Pierre (1992) and King and Vakakis (1993a,b), and is due to a 3:1 internal resonance between modes (n) and ($3n$). *Hence the outlined formulation* [*as well as the invariant-manifold approach of Shaw and Pierre* (1992)] *is only valid for computing nonlinear normal modes which are not in internal resonance.* Results (36) and (37) compute the mode shape of the reference beam 1, during the nonlinear mode oscillation.

The spatial coefficients $a_{21}^{(1)}(s)$ and $a_{23}^{(1)}(s)$ for beam 2 are computed by considering eqns (35) and (33). Satisfying the aforementioned boundary conditions, one obtains the following three solutions for the undetermined coefficient $a_{21}^{(0)}(s_o)$ of the $O(1)$ approximation

$$a_{21}^{(0)}(s_o) = \pm 1, \quad \text{or } a_{21}^{(0)}(s_o) = -16K\sin^2(n\pi s_o)/(9\gamma u_o^{*2}). \tag{38}$$

Solving eqn (35) and the second of eqns (33), one derives the following analytic expressions for coefficients $a_{21}^{(1)}(s)$ and $a_{23}^{(1)}(s)$:

$$a_{21}^{(1)}(s) = C\sin(n\pi s) - \frac{6\gamma u_o^{*2}(n^4\pi^4+k)\,a_{21}^{(0)3}(s_o)}{2560n^4\pi^4(9n^4\pi^4-k)\sin^3(n\pi s_o)}\sin(3n\pi s), \tag{39}$$

$$a_{23}^{(1)}(s) = \frac{a_{11}^{(1)''''}(s_o)\,a_{21}^{(0)}(s_o)+K(1-a_{21}^{(0)2}(s_o))}{6u_o^{*2}(n^4\pi^4+k)\sin(n\pi s_o)}\sin(n\pi s) + \frac{\gamma a_{21}^{(0)3}(s_o)}{32(9n^4\pi^4-k)\sin^3(n\pi s_o)}\sin(3n\pi s). \tag{40}$$

Coefficient $a_{23}^{(1)}(s)$ is completely determined at this order of approximation, whereas the expression for $a_{21}^{(1)}(s)$ contains the yet undetermined coefficient C. This coefficient can only be determined by considering $O(\varepsilon^2)$ terms in eqns (22)–(25). This is due to the fact that no compatibility relations for $a_{21}^{(1)}(s)$ and $a_{23}^{(1)}(s)$ can be formulated; by contrast, $a_{11}^{(1)}(s)$ and $a_{13}^{(1)}(s)$ satisfy the compatibility conditions $a_{11}^{(1)}(s_o) = a_{13}^{(1)}(s_o) = 0$, which were employed to compute expressions (36) and (37). The analytic expression for C is given in the Appendix.

Combining all previous results it is concluded that, *depending on the value of* $a_{21}^{(0)}(s_o)$, *there exist three nonlinear normal modes for the system of Fig. 1.* During a nonlinear normal mode, the modal functions of the two beams are asymptotically approximated as follows:

$$u_1(s,t) = U_1(s,u_o(t)) = \left[\frac{\sin(n\pi s)}{\sin(n\pi s_o)} + \frac{3\varepsilon\gamma(n^4\pi^4+k)\,u_o^{*2}}{320n^4\pi^4(9n^4\pi^4-k)}\left\{\frac{\sin^3(n\pi s)}{\sin^3(n\pi s_o)} - \frac{\sin(n\pi s)}{\sin(n\pi s_o)}\right\}\right]u_o(t) + \frac{\varepsilon\gamma}{8(9n^4\pi^4-k)}\left[\frac{\sin(n\pi s)}{\sin(n\pi s_o)} - \frac{\sin^3(n\pi s)}{\sin^3(n\pi s_o)}\right]u_o^3(t) + O(\varepsilon u_o^5(t),\varepsilon^2) \tag{41}$$

and

$$\begin{aligned} u_2(s,t) = U_2(s,u_o(t)) = &\left[a_{21}^{(0)}(s_o)\frac{\sin(n\pi s)}{\sin(n\pi s_o)} + \varepsilon C\sin(n\pi s) - \frac{6\varepsilon\gamma u_o^{*2}(n^4\pi^4+k)\,a_{21}^{(0)3}(s_o)}{2560n^4\pi^4(9n^4\pi^4-k)\sin^3(n\pi s_o)}\sin(3n\pi s)\right]u_o(t) \\ &+ \left[\frac{\varepsilon\{a_{11}^{(1)''''}(s_o)\,a_{21}^{(0)}(s_o)+K(1-a_{21}^{(0)2}(s_o))\}}{6u_o^{*2}(n^4\pi^4+k)\sin(n\pi s_o)}\sin(n\pi s) + \frac{\varepsilon\gamma a_{21}^{(0)3}(s_o)}{32(9n^4\pi^4-k)\sin^3(n\pi s_o)}\sin(3n\pi s)\right]u_o^3(t) + O(\varepsilon u_o^5(t),\varepsilon^2) \end{aligned} \tag{42}$$

where $a_{21}^{(0)}(s_o)$ is given by eqn (38).

From expressions (41) and (42) it can be easily shown that, when $a_{21}^{(0)}(s_o) = \pm 1$ the modal functions are either equal or opposite in sign, $U_1(s,u_0(t)) = \pm U_2(s,u_0(t))$. In that case the structure vibrates in a *non-localized symmetric or antisymmetric nonlinear normal mode*. When

$$a_{21}^{(0)}(s_o) = -16K\sin^2(n\pi s_o)/(9\gamma u_o^{*2}),$$

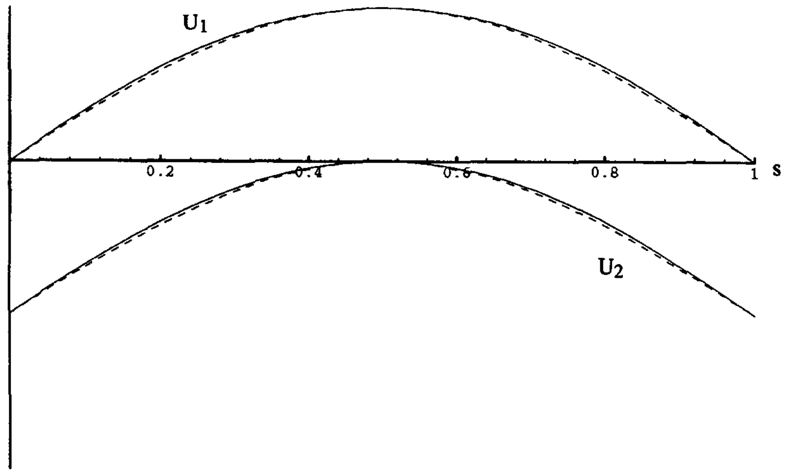

Fig. 2. Asymptotic approximations of the modal functions U_1 and U_2 for the symmetric mode: (———) nonlinear theory ($\varepsilon \neq 0$); (------) linear theory ($\varepsilon = 0$).

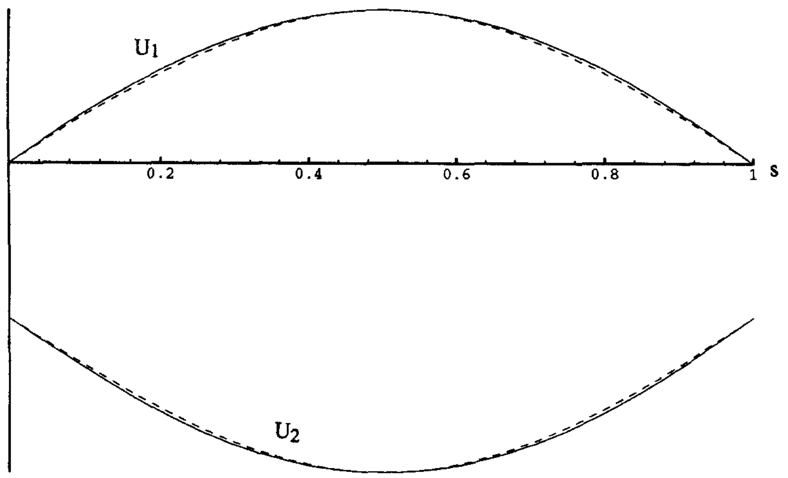

Fig. 3. Asymptotic approximations of the modal functions U_1 and U_2 for the antisymmetric mode: (———) nonlinear theory ($\varepsilon \neq 0$); (------) linear theory ($\varepsilon = 0$).

and the ratio $(K/\gamma u_o^{*2}) \ll 1$, the modal function of beam 1 is much greater in magnitude than that of beam 2; in the limit $(K/\gamma u_o^{*2}) \to 0$, it can be shown that $U_1(s, u_0(t)) \to O(1)$ and $U_2(s, u_o(t)) \to O(\varepsilon u_o^5(t), \varepsilon^2)$. *It is concluded that, as the ratio* $(K/\gamma u_o^{*2})$ *tends to zero, criterion* (4) *of the general analysis is satisfied and the nonlinear normal mode becomes strongly localized to the reference beam* 1. Parameter K is the coefficient of the coupling stiffness, whereas product (γu_o^{*2}) is indicative of the strength of the nonlinearities of the elastic foundations. Hence, the previous analysis shows that *when the ratio of the coupling over nonlinear terms is small, the system possesses a strongly localized nonlinear normal mode.* This conclusion is in agreement with the findings of previous theoretical and numerical works on nonlinear localization (Vakakis and Centinkaya, 1992; King and Vakakis, 1993a), where discrete or discretized periodic systems were considered. Note, that due to the symmetry of the structure under consideration, an additional strongly localized mode exists, involving localization of the free vibration in beam 2. This mode can be studied by considering beam 2 to be the reference substructure. The corresponding localized modal functions for the second localized mode are computed by setting $a_{21}^{(0)}(s_o) = -16K\sin^2(n\pi s_o)/(9\gamma u_o^{*2})$ and interchanging subscripts 1 and 2 in expressions (42).

In Figs 2 and 3 the asymptotic approximations to the modal functions of the non-localized symmetric and antisymmetric normal modes of the structure are depicted for

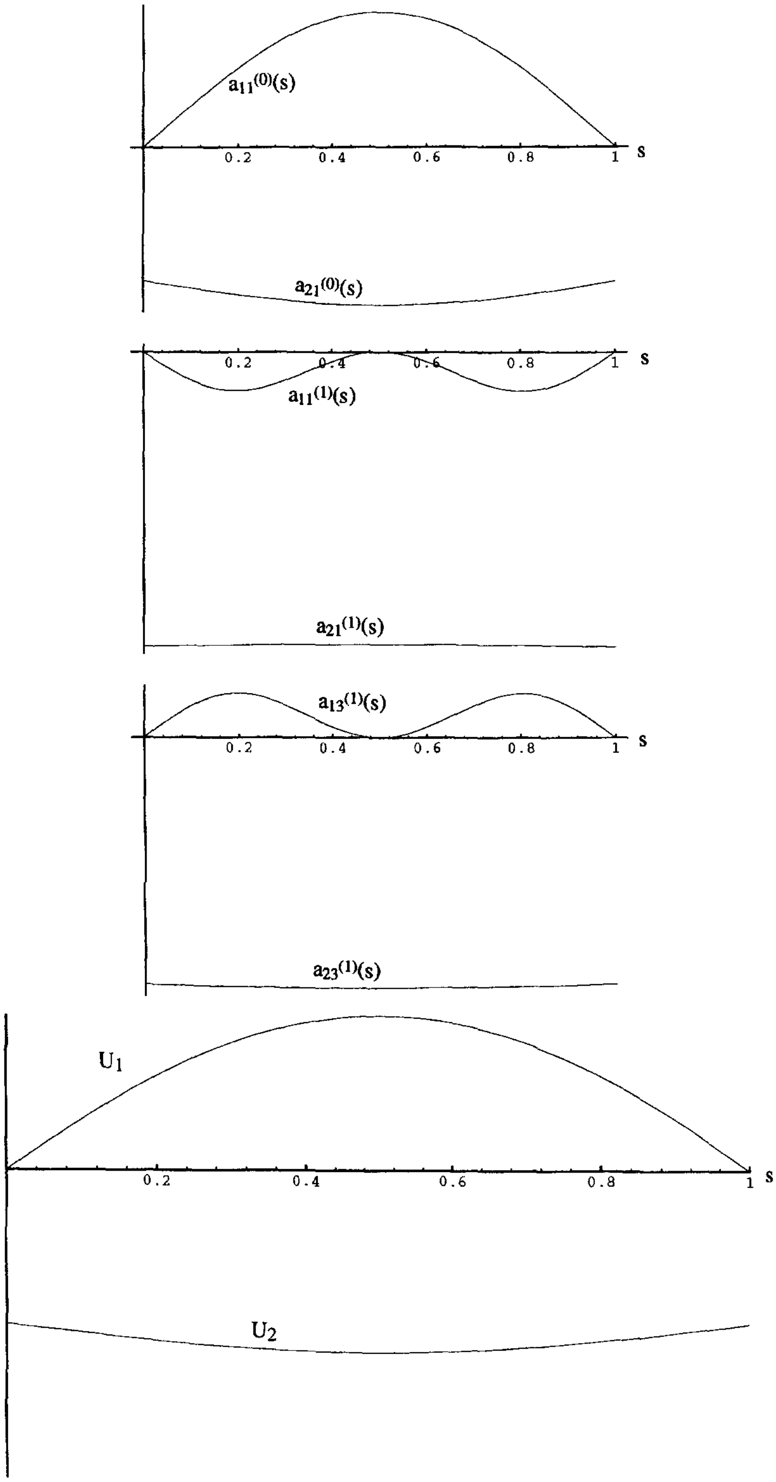

Fig. 4. Asymptotic solutions for the strongly localized mode: (a) spatial coefficients $a_{i1}^{(0)}(s), a_{i1}^{(1)}(s)$ and $a_{i3}^{(1)}(s), i = 1, 2$, (b) modal functions U_1 and U_2.

$n = 1$, $s_0 = 0.5$, $u_o^* = 1.0$, $K = 25$, $k = 850$, $\gamma = 100$ and $\varepsilon = 0.5$. For comparison purposes the respective linearized modes corresponding to $\varepsilon = 0$ are also shown. In Fig. 4a the spatial coefficients $a_{i1}^{(0)}(s)$, $a_{i1}^{(1)}(s)$, and $a_{i3}^{(1)}(s)$, $i = 1, 2$ for the strongly localized mode are depicted for $n = 1$, $s_0 = 0.5$, $u_o^* = 1.0$, $K = 10$, $k = 850$, $\gamma = 100$, and $\varepsilon = 0.5$, whereas in Fig. 4b the complete asymptotic approximations U_1 and U_2 to the localized modal functions are presented. Note that, during the strongly localized mode, the free periodic oscillation of reference beam 1 is of much larger amplitude than that of the non-reference beam 2. *Hence, when the system oscillates in a strongly localized mode, the energy of the motion is mainly confined to only one of the two beams.*

Applying formula (16) of the general analysis and assuming that the system oscillates in a strongly localized mode, one obtains the following differential equation governing the motion of the reference point *during nonlinear mode localization*:

$$\ddot{u}_o(t)+(n^4\pi^4+k)\,u_o(t)+\varepsilon\left[\frac{3\gamma u_o^{*2}(n^4\pi^4+k)(4\sin^2 n\pi s_o-3)}{16(9n^4\pi^4-k)\sin^2 n\pi s_o}+K\left\{1+\frac{16K\sin^2(n\pi s_o)}{(9\gamma u_o^{*2})}\right\}\right]u_o(t)$$
$$+\varepsilon\frac{\gamma(15n^4\pi^4-2(n^4\pi^4+k)\sin^2 n\pi s_o)}{2(9n^4\pi^4-k)\sin^2 n\pi s_o}u_o^3(t)+O(\varepsilon u_o^5(t),\varepsilon^2). \quad (43)$$

For a specified set of initial conditions, the solution of eqn (43) can be expressed in terms of elliptic functions. Assuming that $u_o(0)=u_o^*$, $u_{ot}(0)=0$, i.e. that the system is initiated from a position of maximum potential energy, the solution of eqn (43) is given by

$$u_o(t)=u_o^* cn\,(p_1 t,k_1),\qquad \beta>0$$
$$u_o(t)=u_o^* sn\,(p_2 t+K(k_2),k_2),\qquad -1<\beta u_o^{*2}/\alpha<0, \quad (44)$$

where $cn\,(\cdot,\cdot)$ is the elliptic cosine, with argument p_1 and modulus k_1 given by

$$p_1=[\alpha+\beta u_o^{*2}]^{1/2},\quad k_1^2=\beta u_o^{*2}/2p_1^2, \quad (45a)$$

$sn\,(\cdot,\cdot)$ is the elliptic sine, with argument p_1 and modulus k_1 given by

$$p_2=[\alpha+\beta u_o^{*2}/2]^{1/2},\quad k_2^2=-\beta u_o^{*2}/2p_2^2, \quad (45b)$$

and α and β are the linear and cubic coefficients of the modal oscillator (43), respectively

$$\alpha=n^4\pi^4+k+\frac{3\varepsilon\gamma u_o^{*2}(n^4\pi^4+k)(4\sin^2 n\pi s_o-3)}{16(9n^4\pi^4-k)\sin^2 n\pi s_o}+\varepsilon K\left[1+\frac{16K\sin^2(n\pi s_o)}{(9\gamma u_o^{*2})}\right]$$
$$\beta=\frac{\varepsilon\gamma u_o^{*2}(15n^4\pi^4-2(n^4\pi^4+k)\sin^2 n\pi s_o)}{2(9n^4\pi^4-k)\sin^2 n\pi s_o}. \quad (45c)$$

The frequency of oscillation ω of the strongly localized mode is then computed as

$$\omega=\omega(u_o^*)=\pi p_1/2K(k_1),\quad \beta>0$$
$$\omega=\omega(u_o^*)=\pi p_2/2K(k_2),\quad -1<\beta u_o^{*2}/\alpha<0, \quad (46)$$

where $K(\cdot)$ is the complete elliptic integral of the first kind. Employing eqns (44)–(45), the time responses of the two beams can be computed using the previously derived expressions (41) and (42).

The stability of the computed localized mode is now examined by performing a numerical study similar to the one first introduced in King and Vakakis (1993b). Denoting by $\tilde{u}_i(s,t)$ the motion of beam i during mode localization [expressions (41)–(42) and (44)–(45)], small perturbations in space and time are introduced in the following form:

$$u_i(s,t) = \tilde{u}_i(s,t) + \varepsilon\xi_i(s,t), \quad i = 1,2. \tag{47}$$

Substituting eqn (47) into the equations of motion (17), and retaining only terms of $O(\varepsilon)$, the following set of variational equations results:

$$\xi_{itt} = -\xi_{issss} - k\,\xi_i - 3\varepsilon\gamma\, U_i^{(0)2}\xi_i - \varepsilon K(\xi_i - \xi_{i-1}) + O(\varepsilon^2), \quad i = 1,2, \tag{48}$$

where $\xi_3 \equiv \xi_1$, and $U_i^{(0)}(s,u_o) = a_{i1}^{(0)}(s)u_o(t)$. Equations (48) are linear partial differential equations with time-periodic coefficients. Their solutions are approximated by expansions in terms of orthogonal polynomials as follows:

$$\xi_i(s,t) = \sum_{m=1}^{\infty} \alpha_{im}(t)\,P_m(s), \tag{49}$$

where $\{P_m(s)\}_{m=1}^{\infty}$ is a complete family of orthogonal polynomials, satisfying orthogonality relations of the form, $\int_0^1 P_m(s)\,P_n(s)\,w(s)\,ds = h_n^2\delta_{mn}$, where $w(s)$ is a weighting function and δ_{mn} is Kronecker's symbol. Polynomials $P_m(s)$ need not satisfy the boundary conditions at $s = 0, 1$, but the numerical convergence of the stability analysis is greatly improved if they do. Selecting $P_m(s) = \sin(m\pi s)$, $w(s) = 2$, $h_n = 1$, substituting eqn (49) into eqn (48), premultiplying both sides by $(w(s)\,P_q(s))$, $n = 1, 2, \ldots$, integrating from $s = 0$ to $s = 1$, and neglecting terms of $O(\varepsilon^2)$ or higher, an infinite set of linear ordinary differential equations with time-periodic coefficients is obtained:

$$\alpha_{1qtt} + (k + q^2\pi^2)\,\alpha_{1q} + \varepsilon K(\alpha_{1q} - \alpha_{2q}) + \varepsilon \sum_{m=1}^{\infty} S_{1qm}(t)\,\alpha_{1m} = 0$$

$$\alpha_{2qtt} + (k + q^2\pi^2)\,\alpha_{2q} + \varepsilon K(\alpha_{2q} - \alpha_{1q}) + \varepsilon \sum_{m=1}^{\infty} S_{2qm}(t)\,\alpha_{2m} = 0, \quad q = 1, 2, \ldots, \tag{50}$$

where $S_{iqm}(t) = 6\gamma\, u_o^2(t) \int_0^1 a_{i1}^{(0)2}(s)\sin(q\pi s)\sin(m\pi s)\,ds$ for $u_0(t)$ given by eqn (44), and the quantity $a_{i1}^{(0)}(s)$ was computed previously. For the numerical calculations the infinite set (50) is truncated to $2N$ equations by considering only the first N terms in eqn (49), and Floquet analysis is applied to the truncated system (Nayfeh and Mook, 1984). The Floquet matrix of eqns (50) is computed, and the stability of the variational equations (50) is determined by computing the eigenvalues of the Floquet matrix: eigenvalues greater in modulus than unity indicate orbital instability, whereas eigenvalues of unit modulus correspond to neutral stability. Neutral stability does not necessarily imply stability, but *combined with the fact that the system under consideration is conservative*, neutral stability implies small bounded oscillations of the perturbations $\varepsilon\xi_i(s,t)$ close to the steady-state solutions $\tilde{u}_i(s,t)$. Since truncation of the infinite series (49) was performed, a convergence study must be performed on the eigenvalues of the Floquet matrix. Stability or instability of equations (50) implies orbital stability or instability for the localized normal mode under consideration.

Numerical computations of the Floquet matrix of equations (50) were performed. A parametric study was performed in order to achieve convergence of the eigenvalues of the Floquet matrix. The numerical computations indicate that the Floquet matrix possesses pairs of complex conjugate eigenvalues of unit modulus, indicating orbital (neutral) stability for the nonlinear localized mode. *Hence, it is shown that the detected strongly localized mode is stable and, thus, physically realizable.*

4. DISCUSSION

A new analytical method for studying localization of nonlinear normal modes in a class of periodic structural assemblies was presented. The outlined methodology was based on a direct asymptotic analysis of the governing nonlinear partial differential equations of motion, and did not involve Gallerkin expansions or any other discretization scheme. As a result, the analysis provided nonlinear corrections to the modal amplitudes *and* to the localized mode shapes of the structural elements. The orbital stability of the computed localized modes was numerically studied by expanding the variational equations in terms of orthogonal polynomials and performing Floquet analysis on the resulting set of coupled ordinary differential equations with time-periodic coefficients. The described asymptotic analysis can be carried out up to any desired order of approximation, although for higher approximations the associated analytical computations become cumbersome and resort to computer algebra is required.

Application of the method was given by considering a system of two weakly coupled simply supported beams resting on nonlinear elastic foundations. It was shown that, as the ratio of coupling over nonlinear terms tends to zero, strong nonlinear mode localization occurs in the system, where the free oscillation was mainly confined to only one of the two beams. Moreover, the strongly localized mode was found to be orbitally stable, and, thus, physically realizable. The basic generating mechanism of nonlinear mode localization is the dependence of the frequency of the free nonlinear oscillation on the amplitude of the motion (Vakakis and Cetinkaya, 1992). When the two beams of the coupled assembly are initiated at differing initial amplitudes, their frequencies during free oscillation differ, so that a *frequency disorder* occurs in the system. The effect of this frequency disorder is to confine the amplitude of motion of each beam close to its initial state, and to give rise to nonlinear localization. In the application considered in this work only strong mode localization was detected. Periodic systems with additional number of substructures are expected to possess strongly *and* weakly localized nonlinear modes. Moreover, not all of the localized modes are expected to be orbitally stable.

The nonlinear mode localization phenomenon can be employed for the design of periodic structures with improved passive or active motion confinement properties. Indeed, a direct link between nonlinear mode localization and passive motion confinement of disturbances generated by external impulses was found in Vakakis (1993). In that work it was shown that, if a structure possesses strongly localized modes, motions generated by an external transient disturbance remain passively confined close to the point of application of the disturbance, instead of "spreading" through the entire structure. Clearly, if the inherent dynamics of a structure lead to passive confinement of externally induced motions, this structure is expected to be more amenable to active or passive vibration suppression than a structure with no such motion confinement properties.

Acknowledgements—This work was supported by an NSF Graduate Student Fellowship, and by NSF grant no. MSS 92-07318. Dr Devendra Garg is the grant monitor.

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APPENDIX: ANALYTIC EVALUATION OF COEFFICIENT C OF EXPRESSION (39)

Retaining $O(\varepsilon^2)$ terms in eqns (22)–(25) and imposing the corresponding $O(\varepsilon^2)$ boundary conditions, one obtains the following analytic expression for C:

$$\begin{aligned} C = {} & \left[\Lambda_1 - K(1+a_{21}^{(0)}(s_o)) - \frac{9\gamma u_o^{*2} a_{21}^{(0)2}(s_o)}{4\sin^2(n\pi s_o)}\right]^{-1} \Bigg\{\Gamma_1\Bigg[a_{21}^{(0)}(s_o)\Lambda_1 - K(1+a_{21}^{(0)}(s_o)) \\ & - \frac{9\gamma u_o^{*2} a_{21}^{(0)}(s_o)}{4\sin^2(n\pi s_o)}\Bigg] + \Gamma_2\left[\frac{3\gamma u_o^{*2} a_{21}^{(0)2}(s_o)}{4\sin^2(n\pi s_o)}(1-a_{21}^{(0)4}(s_o))\right] \\ & + \Gamma_3 u_o^{*2}\left[-3\Lambda_1 + K(1+a_{21}^{(0)}(s_o)) + \frac{9\gamma u_o^{*2} a_{21}^{(0)2}(s_o)}{4\sin^2(n\pi s_o)}\right] \\ & + \Gamma_4 u_o^{*2}\left[3a_{21}^{(0)}(s_o)\Lambda_1 - K(1+a_{21}^{(0)}(s_o)) - \frac{9\gamma u_o^{*2} a_{21}^{(0)}(s_o)}{4\sin^2(n\pi s_o)}\right] \\ & + \Gamma_5\left[\frac{3\gamma u_o^{*4} a_{21}^{(0)2}(s_o)}{4\sin^2(n\pi s_o)}(1-a_{21}^{(0)4}(s_o))\right]\Bigg\}, \end{aligned} \tag{A1}$$

where

$$\begin{aligned} \Lambda_1 &= a_{11}^{(1)''''}(s_o) + u_o^{*2} a_{13}^{(1)''''}(s_o) + \gamma u_o^{*2} + K(1-a_{21}^{(0)}(s_o)), \\ \Gamma_1 &= \left\{\frac{\sin(3n\pi s_o)}{\sin^4(n\pi s_o)}\right\}\left\{\frac{6\gamma(n^4\pi^4+k)u_o^{*2}}{2560 n^4\pi^4(9n^4\pi^4-k)}\right\}, \\ \Gamma_2 &= \frac{-6\gamma(n^4\pi^4+k)u_o^{*2}}{2560\, n^4\pi^4(9n^4\pi^4-k)\sin^3(n\pi s_o)}, \\ \Gamma_3 &= \Gamma_4 a_{21}^{(0)}(s_o) + \frac{K(1-a_{21}^{(0)2}(s_o))}{6u_o^{*2}(n^4\pi^4+k)\sin(n\pi s_o)} \\ \Gamma_4 &= \left\{\frac{\sin(3n\pi s_o)}{\sin^4(n\pi s_o)}\right\}\left\{\frac{-\gamma}{32(9n^4\pi^4-k)}\right\}, \\ \Gamma_5 &= \frac{\gamma}{32(9n^4\pi^4-k)\sin^3(n\pi s_o)}. \end{aligned} \tag{A2}$$

For the symmetric ($a_{21}^{(0)}(s_o) = 1$) and anti-symmetric ($a_{21}^{(0)}(s_o) = -1$) cases, the above solution simplifies to give

$$C = \Gamma_1 = \left\{\frac{\sin(3n\pi s_o)}{\sin^4(n\pi s_o)}\right\}\left\{\frac{6\gamma(n^4\pi^4+k)u_o^{*2}}{2560\, n^4\pi^4(9n^4\pi^4-k)}\right\} \quad \text{(symmetric)}, \tag{A3}$$

$$C = -\Gamma_1 = -\left\{\frac{\sin(3n\pi s_o)}{\sin^4(n\pi s_o)}\right\}\left\{\frac{6\gamma(n^4\pi^4+k)u_o^{*2}}{2560\, n^4\pi^4(9n^4\pi^4-k)}\right\} \quad \text{(anti-symmetric)}. \tag{A4}$$

The localized solution, obtained by substituting $a_{21}^{(0)}(s_o) = -16K\sin^2(n\pi s_o)/(9\gamma u_o^{*2})$ into eqn (A1), was evaluated using *Mathematica*. The resulting expression can be easily manipulated symbolically, but is too lengthy to be included here.